Radiation by weakly nonlinear shallow-water solitons due to higher-order dispersion

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Nonlinear asymptotic equations for shallow-water waves, with account of high-order dispersion and surface tension [generalized Boussinesq system (GBS) and generalized Korteweg–de Vries (GKdV) equation] are derived. Regular expansions of these equations in powers of a dispersion parameter lead to different types of already used KdV-type equations, in particular to fifth- and higher-order KdV equations. It is shown that the fifth-order KdV equation describes in a good approximation the shape of a shallow-water soliton, but is insufficient for the consistent description of soliton resonant radiation. The latter is caused by the resonant interaction between the soliton and a plane wave with the phase velocity equal to the soliton velocity. It is shown that the resonant radiation can be correctly described only by equations that take into account dispersive effects to all orders in a region beyond the soliton. The GKdV equation possesses this property and a theory of the soliton resonant radiation, based on the GKdV equation, is developed. It is shown that an account for the full dispersion law for the radiation significantly changes the results obtained earlier by means of the fifth-order KdV equation. A soliton damping caused by its resonant radiation is investigated by means of the GKdV equation. [S1063-651X(98)03710-6]

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I. INTRODUCTION

There are a number of regular procedures for deriving approximate partial differential equations from the full sets of equations describing nonlinear dispersive systems (e.g., [1–5], etc.). For weakly nonlinear waves in shallow water they lead, in the lowest approximation, to the Boussinesq and Korteweg–de Vries (KdV) equations. Higher-order corrections to the Boussinesq and KdV equations permit one to describe different types of perturbation effects. Among them there are effects described by *linear* high-order dispersive terms. A typical example, which has attracted significant attention in recent years, is the fifth-order KdV equation (e.g., Refs. [6–11]). It contains, in addition to the third-order derivative, a linear fifth derivative term that can lead to a new effect: the resonant soliton radiation [8–11]. If the coefficient before the fifth derivative is small, the amplitude of the radiation is exponentially small and, therefore, this effect belongs to the phenomena that are ''beyond all power orders of the perturbation parameter'' [12]. Thus, it cannot be described by a perturbation theory, based on the expansion in powers of the perturbation parameters. Despite the smallness of the resonant soliton radiation, it is very interesting because it may lead to a significant soliton attenuation at large distances and has a rather general nature: it may take place in other high-order dispersive systems, e.g., systems described by higher-order nonlinear Schrödinger equations [13,10,14–16] in one and higher space dimensions as well as by other equations [17–21]. In higher dimensions, the soliton radiation makes a serious impact on self-focusing and collapse [14,18].

The fifth-order KdV equation [6–11] should be considered a result of a proper expansion in powers of the dispersive and nonlinearity parameters, ν and λ, defined below. For the problem of soliton radiation, the fifth-derivative term (with a small coefficient) is mostly significant in the region beyond the soliton. Indeed, the resonant radiation wave number, which basically determines the properties of radiation, is calculated by applying the fifth-order KdV equation to this domain. In the soliton domain, the fifth derivative term (as well as nonlinear terms, which are of higher-order than in the KdV equation) plays the role of small perturbations that mainly determine the soliton deformation; it can be neglected in the lowest approximation. Similar remarks are relevant to seventh- and higher-order approximations.

However, as will be shown below, the fifth- and higher-order KdV-type equations are insufficient for a *quantitative* description of radiation, even at small dispersion parameter ν. Indeed, to justify their applicability to the soliton radiation, one must require $\nu k^2 \ll 1$, where k is the wave number of radiation. On the other hand, from the fifth-order KdV equation it follows that $k^2 \sim \nu^{-1}$, i.e., $\nu k^2 \sim 1$ [6–11]. An account of the seventh-order dispersion does not improve the situation. Due to that a quantitative theory of the soliton radiation, caused by the higher-order dispersion, cannot be based on the expansion in powers of the dispersion parameter ν.

The development of the quantitative theory of soliton resonant radiation is the main goal of the present paper. To achieve it, we first derive equations generalizing the Boussinesq and KdV equations for shallow-water waves in such a way that the exact dispersive law for the gravity-capillary waves follows from them. The application of such equations [we call them generalized Boussinesq (GB) and generalized KdV (GKdV) equations] to the soliton radiation gives correct wave numbers and amplitudes of the radiation. Formal expansions of GB and GKdV equations in the powers of dispersion parameter ν give, to first order, the classical Boussinesq and KdV equations (or similar equations, having the same accuracy) and, to higher orders, the fifth-order KdV, seventh-order KdV equations, etc.

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The paper is organized as follows. In Secs. II and III we derive the generalized Boussinesq system and the GKdV equation and, on their basis, other types of KdV equations. A theory of the resonant soliton radiation is described in Sec. IV using the GKdV equation. Then the results are compared with those following from the fifth-order KdV equation. We also compare the obtained equations with those based on the reductive perturbation method [1] and its recent improvements [4]. The soliton attenuation caused by the radiation is studied in Sec. V and a short conclusion is given in Sec. VI.

II. GENERALIZED BOUSSINESQ SYSTEM

We start with the equations for potential gravity-capillary waves in an incompressible fluid with the density ρ:

$$\nabla^2\Phi=0, \tag{2.1}$$

$$\left(\frac{\partial\Phi}{\partial Z}\right)_{Z=-h}=0, \tag{2.2}$$

$$\left(\frac{\partial Z_0}{\partial T}+\frac{\partial Z_0}{\partial X}\frac{\partial\Phi}{\partial X}+\frac{\partial Z_0}{\partial Y}\frac{\partial\Phi}{\partial Y}-\frac{\partial\Phi}{\partial Z}\right)_{Z=Z_0}=0, \tag{2.3}$$

$$\left[\frac{\partial\Phi}{\partial T}+\frac{1}{2}(\nabla\Phi)^2+gZ+\frac{p-p_0}{\rho}\right]_{Z=Z_0}=0, \tag{2.4}$$

where $Z_0=Z_0(X,Y,T)$ is the equation of the fluid free surface, $\Phi=\Phi(X,Y,Z;T)$ is the velocity potential, $\nabla\Phi=V$. The equilibrium surface is the plane $Z=0$ and the bottom is at $Z=-h$.

The pressure at the free surface $Z=Z_0$ does not coincide with the external pressure p_0 if there is a surface tension. Generally,

$$p=p_0-\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\equiv p_0-\alpha\nabla[(1+|\nabla Z_0|^2)^{-1/2}\nabla Z_0], \tag{2.5}$$

where $R_{1,2}$ are the principal radii of curvature of the free surface and α is the coefficient of surface tension.

Consider long (with respect to the depth) and small amplitude waves assuming, therefore, a smallness of the following two parameters:

$$\nu=\frac{1}{3}\left(\frac{h}{L}\right)^2, \quad \lambda=\frac{a}{h}, \tag{2.6}$$

where L and a are scales of wavelength and wave amplitude.

We introduce the dimensionless quantities

$$x=X/L, \quad y=Y/L, \quad z=Z/L, \quad t=c_0T/L, \tag{2.7a}$$

$$\zeta=Z_0/a, \quad \tilde{\Phi}=(c_0/agL)\Phi(x,y,z;t), \tag{2.7b}$$

where $c_0=\sqrt{gh}$. Then the solution to Eq. (2.1) with boundary condition (2.2) can be written as

$$\tilde{\Phi}=\frac{1}{(2\pi)^2}\int\int A(\mathbf{k},t)\cosh[k(z+\tilde{h})]\exp(i\mathbf{k}\cdot\mathbf{r})d^2\mathbf{k}, \tag{2.8}$$

$$\tilde{\Phi}_z(x,y,z)\equiv\partial_z\tilde{\Phi}(x,y,z)=\frac{1}{(2\pi)^2}\int\int kA(\mathbf{k},t)\sinh[k(z+\tilde{h})]\times\exp(i\mathbf{k}\cdot\mathbf{r})d^2\mathbf{k}, \tag{2.9}$$

where

$$\tilde{h}=h/L=\sqrt{3\nu} \tag{2.10}$$

($\tilde{h}$ is the dimensionless unperturbed depth) and the function $A(\mathbf{k},t)$ is determined by other equations and initial conditions. From Eq. (2.9) it is easily seen that condition (2.2) is satisfied.

Observing that $Z=Z_0$ corresponds to $z_0=\lambda\sqrt{3\nu}\zeta$ and introducing the function ϕ,

$$\phi(x,y,t)=\tilde{\Phi}(x,y,0,t), \tag{2.11}$$

we have from Eqs. (2.8) and (2.9)

$$\tilde{\Phi}(x,y,z_0,t)=\phi(x,y,t)+\sqrt{3\nu}\lambda\zeta\tilde{\Phi}_z(x,y,0,t)+O(\lambda^2\nu), \tag{2.12}$$

$$\tilde{\Phi}_z(x,y,0,t)=\hat{k}\tanh(\sqrt{3\nu}\hat{k})\phi(x,y,t), \tag{2.13}$$

$$\tilde{\Phi}_z(x,y,z_0,t)=\hat{k}\tanh(\sqrt{3\nu}\hat{k})\phi(x,y,t)-\lambda\sqrt{3\nu}\zeta\Delta\phi(x,y,t)+O(\lambda^2\nu). \tag{2.14}$$

Now Δ is the two-dimensional Laplacian

$$\Delta=(\partial_x^2+\partial_y^2)\equiv-\hat{k}^2, \tag{2.15a}$$

and an operator $F(\hat{k})$ is defined as

$$F(\hat{k})\Psi(\mathbf{r})=\frac{1}{(2\pi)^2}\int d\mathbf{k}\,F(\mathbf{k})G(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2.15b}$$

where $G(\mathbf{k})$ is the Fourier transform of $\Psi(\mathbf{r})$. A formal expansion gives

$$\hat{k}\tanh(\sqrt{3\nu}\hat{k})=\frac{1}{\sqrt{3\nu}}(3\nu\hat{k}^2)\left(1-\nu\hat{k}^2+\frac{6}{5}\nu^2\hat{k}^4-\cdots\right). \tag{2.16}$$

Thus, the operator in the left-hand side of Eq. (2.16) can be expressed as a power series of the Laplacian (2.15a). [In fact, Eq. (2.16) is an integral operator.] From Eq. (2.16), the expansions of Eq. (2.13) and other expressions follow.

Using Eqs. (2.7) and (2.12)–(2.14), we get from Eq. (2.3) the approximate equation

$$\zeta_t+\lambda\nabla(\zeta\nabla\phi)-(3\nu)^{-1/2}\hat{k}\tanh(\sqrt{3\nu}\hat{k})\phi+O(\lambda\nu,\lambda^2)=0. \tag{2.17}$$

In a similar way, from Eq. (2.4) it follows that

$$\phi_t+\tfrac{1}{2}\lambda(\nabla\phi)^2+\zeta-\nu\sigma\Delta\zeta+O(\lambda\nu)=0, \tag{2.18}$$

where σ is a normalized surface tension coefficient

$$\sigma=3\alpha/\rho gh^2>0. \tag{2.19}$$

It is proportional to the Bond number.

In linear approximation, Eqs. (2.17) and (2.18) become

$$\zeta_t-(3\nu)^{-1/2}\hat{k}\ \tanh(\sqrt{3\nu}\hat{k})\phi+O(\lambda)=0, \qquad (2.20)$$

$$\phi_t+\zeta-\nu\sigma\Delta\zeta+O(\lambda)=0. \qquad (2.21)$$

Eliminating ϕ from Eq. (2.20), we have

$$\zeta_{tt}+(3\nu)^{-1/2}\hat{k}\ \tanh(\sqrt{3\nu}\hat{k})(1-\nu\sigma\Delta)\zeta+O(\lambda)=0. \qquad (2.22)$$

Looking for a plane-wave solution to Eq. (2.22)

$$\zeta\propto\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t), \qquad (2.23)$$

we obtain the linear dispersion equation in dimensionless variables

$$\omega^2=(3\nu)^{-1/2}k(1+\nu\sigma k^2)\tanh(\sqrt{3\nu}k). \qquad (2.24)$$

Passing to dimensional variables Ω and $\mathbf{K}$,

$$\omega=(L/c_0)W, \quad \mathbf{k}=L\mathbf{K}, \qquad (2.25)$$

we have

$$W^2=gK\ \tanh(hK)[1+(1/3)\sigma h^2K^2], \qquad (2.26)$$

which is the well-known exact linear dispersion equation for the gravity-capillary waves in an inviscid incompressible fluid.

Now, consider the opposite case of nonlinear, but nondispersive waves. Assuming in Eqs. (2.17) and (2.18) that $\nu\to 0$ and taking into account Eq. (2.16), we have

$$\zeta_t+\lambda\nabla(\zeta\nabla\phi)+\Delta\phi=0, \qquad (2.27)$$

$$\phi_t+\tfrac{1}{2}\lambda(\nabla\phi)^2+\zeta=0. \qquad (2.28)$$

Introducing the quantity

$$\eta=1+\lambda\zeta, \qquad (2.29)$$

we obtain from Eq. (2.27)

$$\eta_t+\lambda\boldsymbol{\nabla}\cdot(\eta\boldsymbol{\nabla}\phi)=0. \qquad (2.30)$$

Expressing η through dimensional variables we have

$$\eta=(Z_0+h)/h=H/h, \qquad (2.31)$$

where $H=Z_0+h$ is the full height of a point on the fluid surface. Thus, η is the dimensionless full height. In dimensional variables, Eq. (2.30) takes the form

$$\partial_T H+\mathrm{div}(H\mathbf{V})=0, \qquad (2.32)$$

where

$$\mathbf{V}=\mathrm{grad}\ \Phi(X,Y,0,T). \qquad (2.33)$$

Equation (2.28) can be transformed to

$$\partial_T\mathbf{V}+(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}+\frac{1}{2H}\boldsymbol{\nabla}(gH^2)=\mathbf{0}, \qquad (2.34)$$

where $\boldsymbol{\nabla}=(\partial_X,\partial_Y)$. Equations (2.32) and (2.34) are the well-known equations of shallow water in the nondispersive limit; they coincide with the equations describing a two-dimensional flow of an ideal gas with the adiabatic index $\gamma=2$; Eq. (2.32) is the continuity and Eq. (2.34) is the Euler equation (H plays the role of density and $p=gH^2/2$ is a "pressure"). Respectively, Eq. (2.30) is the continuity equation in dimensionless variables and, therefore, $\boldsymbol{\nabla}\phi(x,y,t)$ is the dimensionless velocity that describes the horizontal mass transfer. Taking into account the dispersion, we come to Eq. (2.17), which is not a continuity equation, and thus $\boldsymbol{\nabla}\phi(x,y,t)$, with the account of dispersion, does not represent the effective velocity that describes the horizontal mass transfer.

To define an effective horizontal velocity determining the mass transfer, we introduce the renormalized potential $\psi(x,y,t)$ defined by

$$\phi=\sqrt{3\nu}\hat{k}\ \coth(\sqrt{3\nu}\hat{k})\psi. \qquad (2.35)$$

The expansion

$$\sqrt{3\nu}\hat{k}\ \coth(\sqrt{3\nu}\hat{k})=1+\nu\hat{k}^2-\frac{\nu^2}{5}\hat{k}^4+\cdots \qquad (2.36)$$

is again a power series of the Laplacian (2.15a). Neglecting the terms of order $\lambda\nu$, we have instead of Eq. (2.17)

$$\zeta_t+\lambda\nabla(\zeta\nabla\psi)+\Delta\psi=0. \qquad (2.37)$$

Introducing the dimensionless renormalized velocity as

$$\mathbf{v}=\boldsymbol{\nabla}\psi, \qquad (2.38)$$

and using the full height (2.29), we have from Eq. (2.37) the continuity equation

$$\eta_t+\lambda\boldsymbol{\nabla}\cdot(\eta\mathbf{v})=0, \qquad (2.39)$$

which shows that $\mathbf{v}$ is the effective horizontal velocity determining the mass transfer with the used accuracy (terms with $\lambda\nu$ are neglected). In a similar way Eq. (2.18), with account of Eq. (2.38), can be transformed to

$$\sqrt{3\nu}\hat{k}\ \coth(\sqrt{3\nu}\hat{k})\mathbf{v}_t+\lambda(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\boldsymbol{\nabla}\zeta-\nu\sigma\Delta\boldsymbol{\nabla}\zeta=\mathbf{0}. \qquad (2.40)$$

Equations (2.39) and (2.40) will be called the generalized Boussinesq System (GBS). Taking into account two first terms in the expansion (2.36), we come to the equation, equivalent to the classical Boussinesq equation [2,3]

$$(1-\nu\Delta)\mathbf{v}_t+\lambda(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\boldsymbol{\nabla}\zeta-\nu\sigma\Delta\boldsymbol{\nabla}\zeta=0. \qquad (2.41)$$

For a more convenient comparison of Eq. (2.40) with Eq. (2.41), we define the operator

$$E(\nu\hat{k}^2)=1-\nu\Delta-\sqrt{3\nu}\hat{k}\ \coth(\sqrt{3\nu}\hat{k}), \qquad (2.42)$$

which has the expansion

$$E(\nu\hat{k}^2)=(1/5)\nu^2\Delta^2+(2/35)\nu^3\Delta^3+\cdots. \qquad (2.43)$$

Then Eq. (2.40) can be written as

$$[1-\nu\Delta-E(\nu\hat{k}^2)]\mathbf{v}_t+\lambda(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\boldsymbol{\nabla}\zeta-\nu\sigma\Delta\boldsymbol{\nabla}\zeta+O(\lambda\nu)=0. \tag{2.44}$$

We see that the classical Boussinesq equation follows from Eq. (2.44) when $E(\nu\hat{k}^2)$ is neglected. As long as the terms with $\lambda\nu$ are neglected in Eq. (2.40) and, therefore, in Eqs. (2.41) and (2.44), we can substitute in the first term of Eq. (2.44),

$$\Delta\mathbf{v}=-\boldsymbol{\nabla}\zeta_t+O(\lambda), \tag{2.45}$$

which follows from the continuity equation (2.37) in linear approximation. Then Eq. (2.44) can be written, with the same accuracy, in the form

$$[1-E(\nu\hat{k}^2)]\mathbf{v}_t+\lambda(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\lambda^{-1}\boldsymbol{\nabla}\eta+\nu\boldsymbol{\nabla}(\zeta_{tt}-\sigma\Delta\zeta)+O(\lambda\nu)=0. \tag{2.46}$$

Equation (2.46) can also be called a GB equation. In a similar way one can obtain other GB-type equations, having the same accuracy as Eq. (2.44).

It is easy to see that Eq. (2.22) and the exact linear dispersion equation (2.24) follow also from the linearized GB system.

III. KORTEWEG–dE VRIES–TYPE EQUATIONS

Further simplifications of GBS can be made for one-dimensional waves, when $v=v(x,t)$ and $\rho=\rho(x,t)$. In this case, following an approach of Ref. [3], we shall derive the Korteweg–de Vries–type equations that describe higher-order dispersive effects.

First, neglecting the dispersive terms in Eq. (2.44) and using the full dimensionless depth (2.29), we come to the equation

$$\mathbf{v}_t+\lambda(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\lambda^{-1}\boldsymbol{\nabla}\eta+O(\nu)=0. \tag{3.1}$$

The system of equations (2.39) and (3.1) has a simple wave solution $\eta=\eta(v)$ with

$$\eta(v)=(1+\tfrac{1}{2}\lambda v)^2. \tag{3.2}$$

Then $v=v(x,t)$ satisfies the equation

$$v_t+v_x+\tfrac{3}{2}\lambda v v_x+O(\nu)=0. \tag{3.3}$$

To find an extension of the simple waves for the GBS (quasisimple waves), we assume that [3]

$$\eta(x,t)=\eta(v)+\nu\lambda\varphi(x,t), \tag{3.4}$$

where $\eta(v)$ is the same as in the simple wave at $\nu=0$, i.e., it is given by Eq. (3.2) and $\varphi(x,t)$ is an unknown function. Substituting Eq. (3.4) in Eqs. (2.39) and (2.44), we come to the following equations:

$$v_t+v_x+\tfrac{3}{2}\lambda v v_x+\nu\varphi_t=O(\lambda^2,\lambda\nu), \tag{3.5}$$

$$[1-\nu\partial_x^2-E(\nu\hat{k}^2)]v_t+v_x+\tfrac{3}{2}\lambda v v_x-\nu\sigma\partial_x^3 v+\nu(1-\nu\sigma\partial_x^2)\varphi_x=O(\lambda^2,\lambda\nu). \tag{3.6}$$

From Eqs. (3.5) and (3.6) it follows that

$$\nu(1-\nu\sigma\partial_x^2)\varphi_x-\nu\varphi_t=[\nu\partial_x^2+E(\nu\hat{k}^2)]v_t+\nu\sigma\partial_x^3 v+O(\lambda\nu,\lambda^2), \tag{3.7}$$

where now $\hat{k}=-i\partial_x$.

First, we take into account the dispersive terms of the lowest order, neglecting terms $O(\nu^2)$ and, respectively, $E(\nu\hat{k}^2)$. Then Eq. (3.7) is reduced to

$$\nu(\varphi_x-\varphi_t)=\nu(v_{tx}+\sigma v_{xx})_x+O(\lambda^2,\lambda\nu). \tag{3.8}$$

For the wave, propagating in positive direction, we can write in the same approximation

$$v_t=-v_x+O(\lambda,\nu), \tag{3.9}$$

$$\varphi_t=-\varphi_x+O(\lambda,\nu). \tag{3.10}$$

Combining Eq. (3.8) with Eqs. (3.9) and (3.10), we obtain

$$\varphi=\tfrac{1}{2}(v_{xt}+\sigma v_{xx})+O(\lambda,\nu)=\tfrac{1}{2}(1-\sigma)v_{xt}+O(\lambda,\nu)=-\tfrac{1}{2}(1-\sigma)v_{xx}+O(\lambda,\nu). \tag{3.11}$$

Substituting this in Eq. (3.5), we come to the following three equations, having the same accuracy:

$$v_t+v_x+\tfrac{3}{2}\lambda v v_x-\tfrac{1}{2}\nu(v_t+\sigma v_x)_{xx}=0, \tag{3.12}$$

$$v_t+v_x+\tfrac{3}{2}\lambda v v_x-\tfrac{1}{2}\nu(1-\sigma)v_{txx}=0, \tag{3.13}$$

$$v_t+v_x+\tfrac{3}{2}\lambda v v_x+\tfrac{1}{2}\nu(1-\sigma)v_{xxx}=0 \tag{3.14}$$

[terms $O(\lambda^2,\lambda\nu)$ are neglected]. Each of them describes a nonlinear quasisimple dispersive wave, propagating in the positive direction, to the first order of λ and ν. Equations (3.12) and (3.13) at $\sigma=0$ turn into one equation that had been considered by Peregrine and Benjamin, Bona, and Mahoney [22] while Eq. (3.14) is the well-known Korteweg–de Vries equation.

To find the dispersive corrections to Eqs. (3.12)–(3.14) of arbitrary order, one must add higher-order terms to (3.8)–(3.10). To do this, we first write Eq. (3.7) in the form

$$\nu(1+\nu\sigma\hat{k}^2)\varphi_x-\nu\varphi_t=[\sqrt{3\nu}\,\coth(\sqrt{3\nu}\hat{k})-1]v_t-i\nu\sigma\hat{k}^3 v+O(\lambda\nu,\lambda^2). \tag{3.15}$$

For the wave propagating to the right, we now have

$$v_t=-i\omega(\hat{k})v+O(\lambda),\quad \varphi_t=-i\omega(\hat{k})\varphi+O(\lambda), \tag{3.16}$$

where, according to Eq. (2.24),

$$\omega(\hat{k})=(3\nu)^{-1/4}(1+\nu\sigma\hat{k}^2)^{1/2}[\hat{k}\,\tanh(\sqrt{3\nu}\hat{k})]^{1/2}. \tag{3.17}$$

From Eq. (3.15)–(3.17) it follows that

$$\nu\omega(\hat{k})\varphi=-L(\hat{k})v+O(\lambda\nu,\lambda^2), \tag{3.18}$$

where

$$L(\hat{k})=\omega(\hat{k})-\hat{k}. \quad (3.19)$$

Differentiating Eq. (3.18) with respect to t and then substituting v_t from Eq. (3.16), we have

$$\nu\varphi_t=iL(\hat{k})v+O(\lambda\nu,\lambda^2). \quad (3.20)$$

From Eqs. (3.5) and (3.20) we easily derive the equation

$$v_t+v_x+\tfrac{3}{2}\lambda v v_x+iL(\hat{k})v+O(\lambda\nu,\lambda^2)=0, \quad (3.21)$$

which will be called the generalized KdV equation. In linear approximation it coincides with the first of Eqs. (3.16). Equation (3.21) was proposed earlier [23] as a *model* equation for weakly nonlinear long internal waves in stratified fluids of finite depth. In fact, as is seen from the above derivation, it is a rigorous approximate equation, that can be obtained in a regular way by the multiscale method with the accuracy indicated in Eq. (3.21). In the next section it will be demonstrated that it leads to a consistent theory of the resonant soliton radiation.

Expanding $L(\hat{k})$ in powers of ν we can obtain the dispersion corrections of any order to the KdV equation. In particular, from the expansion

$$iL(\hat{k})=\beta\partial_x^3+\gamma\partial_x^5+\cdots, \quad (3.22)$$

where

$$\beta=\frac{1}{2}\,\nu(1-\sigma),\quad \gamma=\frac{\nu^2}{40}\,[24-5(1+\sigma)^2], \quad (3.23)$$

we come to the fifth-order KdV equation:

$$\partial_t v+(1+\tfrac{3}{2}\lambda v)\partial_x v+\beta\partial_x^3 v+\gamma\partial_x^5 v+O(\lambda\nu,\lambda^2,\nu^3)=0. \quad (3.24)$$

In fact the effect of the resonant soliton radiation, qualitatively, also follows from Eq. (3.24) [8–11]. However, only the full Eq. (3.21) gives a consistent and a complete theory of this effect in shallow fluids. This is because expansion (3.22) is justified if

$$\nu D^{-2}\ll 1, \quad (3.25)$$

where D is the wave characteristic length. In the problem of soliton radiation with wave number k, $D=k^{-1}$ and, as will be shown in the next section, $\nu k^2\sim 1$, which means that condition (3.25) is not satisfied for the soliton radiation.

On the other hand, Eq. (3.24) can be used for the estimation of influence of higher-order dispersion on the soliton width. Indeed, using only the first term on the right-hand side of Eq. (3.22), we come to the KdV equation (3.14), which has a soliton solution of the form

$$v_s=\frac{2(M-1)}{\lambda}\,\mathrm{sech}^2\left[\frac{\kappa_0}{2}\,(x-Mt)\right], \quad (3.26)$$

where κ_0 is the inverse width of unperturbed soliton,

$$\kappa_0^2=\frac{2(M-1)}{\nu(1-\sigma)}. \quad (3.27)$$

Now $D=\kappa_0^{-1}$ and if $M-1$ is sufficiently small, namely,

$$|M-1|\ll(1/6)|1-\sigma|, \quad (3.28)$$

we have

$$3\nu\kappa_0^2\ll 1. \quad (3.29)$$

Thus condition (3.25) is satisfied and the fifth-derivative term in Eq. (3.24) can be considered as small *inside the soliton core*. In fact, Eq. (3.24) cannot be used for the calculation of the full soliton deformation of the next order to Eq. (3.26), because deriving Eq. (3.24) we have neglected nonlinear terms of order $\lambda\nu$ and λ^2 that may be comparable with the fifth-derivative term in the soliton domain. However, the latter term is greater than other corrections at large distances from the soliton center. Therefore, investigating the soliton asymptotics, we can use the linearized Eq. (3.24)

$$\partial_t v+\partial_x v+\beta\partial_x^3 v+\gamma\partial_x^5 v=0 \quad (3.30)$$

with the solution

$$v\propto\exp(-\kappa|x-Mt|), \quad (3.31)$$

where $\kappa>0$ and

$$\kappa^2=\frac{-\beta\pm\sqrt{\beta^2+4(M-1)\gamma}}{2\gamma} \quad (3.32)$$

(the choice of sign depends on sgn β). At condition (3.28), leading to Eq. (3.29), Eq. (3.32) can be reduced to

$$\kappa\approx\kappa_0\left\{1-\frac{(M-1)}{20(1-\sigma)^2}\,[24-5(1+\sigma)^2]\right\}$$

$$\approx\kappa_0\left\{1-\frac{\nu\kappa_0^2}{40(1-\sigma)}\,[24-5(1+\sigma)^2]\right\}, \quad (3.33)$$

where κ_0^2 is given by Eq. (3.27). Evidently, $\kappa_0^2>0$ if

$$\mathrm{sgn}(M-1)=\mathrm{sgn}(1-\sigma). \quad (3.34)$$

The second term in Eq. (3.33) represents a correction to the soliton width caused by higher-order dispersion.

IV. THE RESONANT RADIATION OF WEAKLY NONLINEAR SHALLOW-WATER SOLITONS

Now we return to the full GKdV equation (3.21). Due to Eq. (3.26), it is convenient to write it in the new variables

$$u=\frac{3\lambda}{2(M-1)}\,v, \quad (4.1)$$

$$\xi=\kappa(x-Mt),\quad \tau=(M-1)\kappa t. \quad (4.2)$$

This gives

$$\partial_\tau u-\partial_\xi u+u\,\partial_\xi u+i(M-1)^{-1}\kappa^{-1}L(\hat{k})u=0, \quad (4.3)$$

where, now,

$$\hat{k}=-i\kappa\partial_\xi. \quad (4.4)$$

Then we put

$$u(\tau,\xi)=u_0(\xi)+f(\tau,\xi), \tag{4.5}$$

where

$$u_0=3\,\text{sech}^2(\xi/2) \tag{4.6}$$

describes the soliton (3.26) with the renormalized width (3.33) and $f(\tau,\xi)$ is a small addition. The soliton radiation, if it exists, is described by the asymptotics of $f(\tau,\xi)$ at $|\xi|\gg 1$. Substituting Eq. (4.5) in Eq. (4.3), and taking into account the KdV equation for $u_0(\xi)$, we have the following asymptotic equation for $f(\tau,\xi)$ in the domain beyond the soliton

$$\partial_\tau f-\partial_\xi f+\partial_\xi(u_0 f\ )+i(M-1)^{-1}\kappa^{-1}L(\hat{k})f=-iL_1(\hat{k})u_0, \tag{4.7}$$

where

$$L_1(\hat{k})\approx(M-1)^{-1}\kappa^{-1}L(\hat{k})+\kappa^{-3}\hat{k}^3. \tag{4.8}$$

Beyond the soliton, Eq. (4.7) becomes

$$\partial_\tau f-\partial_\xi f+i(M-1)^{-1}\kappa^{-1}L(\hat{k})f=0. \tag{4.9a}$$

This equation describes the free radiation that is a superposition of plane waves

$$f_q\propto\exp[i(q\xi-\Omega\tau)], \tag{4.9b}$$

where, with account of Eq. (4.4),

$$\Omega(q)=(M-1)^{-1}\kappa^{-1}L(\kappa q)-q. \tag{4.10}$$

It is easy to check that at $\Omega=0$, the corresponding wave number is $q(0)=k/\kappa$, where k satisfies the equation

$$\omega(k)/k=M. \tag{4.11}$$

Here

$$\omega(k)=(3\nu)^{-1/4}(1+\nu\sigma k^2)^{1/2}[k\ \tanh(\sqrt{3\nu}k)]^{1/2}\ \text{sgn}\ k \tag{4.12}$$

is the frequency of the linear wave with the wave number k [cf. (2.24)]. Therefore, Eq. (4.9) at $\Omega=0$ and $q=k/\kappa$ describes a wave that has phase velocity equal to the soliton velocity M. Such a wave must resonantly interact with the soliton and will be called the resonant wave. Respectively k, satisfying Eq. (4.11) at given M, will be called the resonant wave number. The purpose of this section is the study of this resonance effect without expansion of dispersion equation (4.12) in powers of k.

First, we find resonant wave numbers from Eq. (4.11). Using Eq. (4.12), we reduce Eq. (4.11) to the equation

$$\sigma r^2=3(M^2 r\ \coth r-1),\quad r=\sqrt{3\nu}k. \tag{4.13}$$

The real positive roots r of this equation determine the resonant wave numbers for different σ at fixed M, i.e., the function $r(\sigma)$. The inverse function, $\sigma(r)$, immediately follows from Eq. (4.13):

$$\sigma(r)=3r^{-2}(M^2 r\ \coth r-1). \tag{4.14}$$

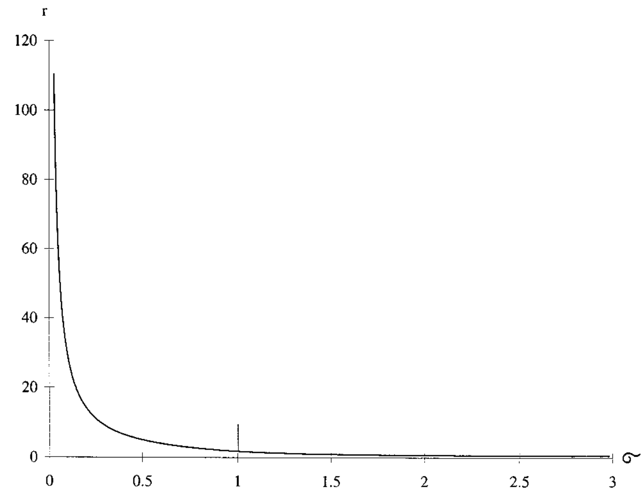

FIG. 1. r vs σ at $M^2=1.1$.

Further investigation of $\sigma(r)$ becomes simpler if one writes it in the form

$$\sigma(r)=\frac{3(M^2-1)}{r\ \tanh r}+3\left(\frac{\coth r}{r}-\frac{1}{r^2}\right). \tag{4.15}$$

The second term in Eq. (4.15) is a decreasing positive function of r at all $-\infty<r<\infty$ and

$$3\left(\frac{\coth r}{r}-\frac{1}{r^2}\right)\approx 1-\frac{r^2}{15}\quad(r\ll 1), \tag{4.16a}$$

$$3\left(\frac{\coth r}{r}-\frac{1}{r^2}\right)\approx\frac{3}{r}\quad(r\gg 1). \tag{4.16b}$$

The maximum of this term is equal to one. The behavior of the full function (4.15) substantially depends on the sign of $M-1$.

At $M>1$, the function $\sigma(r)$ monotonously decreases from $\sigma(0)=\infty$ to $\sigma(\infty)=0$. Therefore, Eq. (4.39) has two real roots, $\pm r(\sigma)$, at

$$M>1,\quad \sigma<1 \tag{4.17}$$

[σ must satisfy condition (3.34)]. The function $\sigma(r)$ is easily tabulated at a given M by means of Eq. (4.14). Then $r(\sigma;M)$ can be found graphically. At $M^2=1.1$, the function $r(\sigma)$ is shown in Fig. 1. At sufficiently small σ, it can be written as

$$r(\sigma)\approx 3M^2/\sigma\quad(\sigma\ll M^2\sim 1). \tag{4.18}$$

Equation (4.18) can be used for estimations with a good accuracy at $\sigma<0.5$. Then from Eq. (3.28) it follows that

$$M-1\ll 0.08. \tag{4.19}$$

At $\sigma>0.5$, the restriction on $M-1$ is even harder. At $M-1>0$ and $0.5<\sigma<1$, the intervals for $r(\sigma;M)$ are rather narrow. Writing Eq. (3.28) in the form $M-1<(C/6)(1-\sigma)$, where $C\ll 1$, and taking as an example $C=1/5$, we have

$$0<M-1<(1/30)(1-\sigma). \tag{4.20}$$

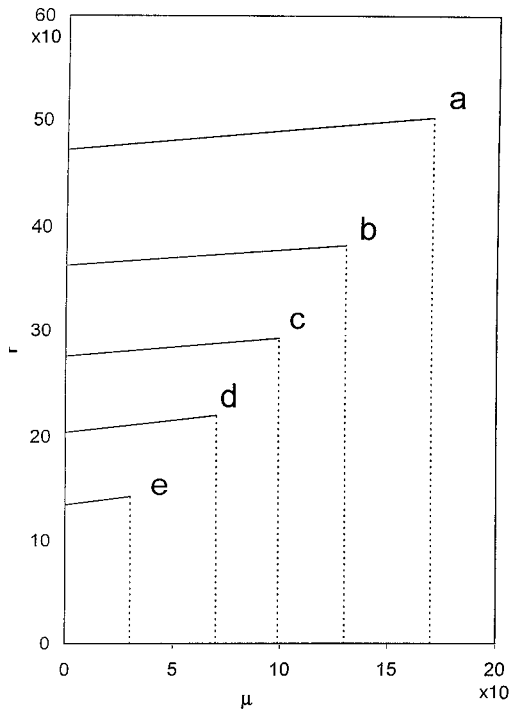

FIG. 2. r vs $\mu=M-1>0$ at different $1>\sigma\geqslant 0.5$. (a) $\sigma=0.5$; (b) $\sigma=0.6$; (c) $\sigma=0.7$; (d) $\sigma=0.8$; (e) $\sigma=0.9$.

On the other hand, from Eq. (4.15) at $M-1\ll 1$ it follows that

$$\mu\approx\frac{1}{2}\left(\frac{\tanh r}{r}-1\right)+\frac{\sigma r\,\tanh r}{6},\qquad \mu=M-1. \tag{4.21}$$

This determines the function $r(\sigma;\mu)$, where one must take into account that μ is restricted by Eq. (3.28). Replacing it, for instance, by Eq. (4.20) and taking $\sigma=0.5$, we have $\mu_{\max}=0.017$. The function $r(\sigma;\mu)$, at some $\sigma\geqslant 0.5$ and μ inside of the interval (4.20), are shown in Fig. 2.

In the other case when solitons exist,

$$M<1,\quad \sigma>1 \tag{4.22}$$

[see Eq. (3.34)], the first term in Eq. (4.15) is negative and

$$\sigma(r)\approx-(1-M^2)r^{-2}\quad (r\ll 1),$$

$$\sigma(r)\approx 3M^2r^{-1}\quad (r\gg 1).$$

In this case, $\sigma(r)$ has a positive maximum. From Eqs. (4.15) and (4.16a) one concludes that $[\sigma(r)]_{\max}<1$. Therefore in the case (4.22), Eq. (4.14) has no real roots $r(\sigma)$. From the above analysis it follows that the resonance is possible in case (4.17) and impossible in case (4.22).

It is easy to check that Eq. (4.13) also has two small imaginary roots $k\approx\pm i\kappa$, i.e., $q\approx\pm i$. Substituting this in Eq. (4.9b) at $\Omega=0$, we see that in this case (4.9b) represents, with the assumed accuracy, the soliton asymptotic behavior at $|\xi|\to\infty$ [cf. Eq. (3.31)].

Now consider the group velocity V_g of the resonant wave. Writing

$$V_g=\frac{d\omega(k)}{dk}\equiv\frac{\omega(k)}{k}\,k\,\frac{d\ln\omega(k)}{dk}=Mr\,\frac{d\ln\omega(r)}{dr}, \tag{4.23}$$

and using Eq. (4.12), we have

$$V_g=M\left[\frac{3(\sigma r^2+1)}{2(\sigma r^2+3)}+\frac{r}{\sinh 2r}\right].$$

Substituting here σ from Eq. (4.14), we come to the following expression for the group velocity:

$$V_g=M\left[1+\frac{M^2-1}{M^2}\,\frac{\tanh r}{r}+\chi(r)\right], \tag{4.24}$$

where

$$\chi(r)=\frac{1}{2}-\frac{\tanh r}{r}+\frac{r}{\sinh 2r}.$$

One can easily check that $\chi(r)$ is a positive and increasing function with

$$\chi(0)=0,\chi(1)=0.014,\quad \chi(2)=0.091,$$

$$\chi(3)=0.183,\quad \chi(\infty)=0.500.$$

We see that

$$V_g>M \tag{4.25a}$$

at all r, and

$$V_g-M\approx M/2\quad (r\gg 1). \tag{4.25b}$$

The obtained wave numbers and group velocities of the resonant soliton radiation essentially differ from those following from the fifth-order KdV equation [8,10,11]. Moreover, the criteria of the soliton radiation are different. As we have seen, from the GKdV equation it follows that solitons can radiate only at condition (4.17). On the other hand, from the fifth-order KdV equation (3.24) it follows that the radiation condition is $\beta\gamma>0$; using Eq. (3.23), we write this as

$$(1-\sigma)[24-5(1+\sigma)^2]>0,$$

which gives, in addition to Eq. (4.17), $\sigma>\sqrt{24/5}-1\approx 1.19$.

Now we shall show that the soliton indeed radiates the resonant wave with the phase velocity equal to M and the group velocity that is larger than M, according to Eq. (4.25). For that, we solve Eq. (4.7) with the initial condition

$$f(0,\xi)=0. \tag{4.26}$$

Thus, we assume that the soliton starts to radiate at $\tau=0$. Performing the Fourier transform of $f(\tau,\xi)$,

$$\varphi(\tau,q)=\int_{-\infty}^{\infty}d\xi\; e^{-iq\xi}f(\tau,\xi) \tag{4.27}$$

and transforming Eq. (4.7), we have the following equation:

$$i\partial_\tau\varphi(\tau,q)-\Omega(q)\varphi(\tau,q)-\frac{q}{2\pi}\int_{-\infty}^{\infty}\varphi_0(q-q')\varphi(\tau,q')dq'$$
$$=L_1(\kappa q)\varphi_0(q), \tag{4.28}$$

where

$$\varphi_0(q)\equiv\int_{-\infty}^{\infty}d\xi\, e^{-iq\xi}u_0(\xi)=12\pi q\ \operatorname{csch}(\pi q) \tag{4.29}$$

and we took into account Eqs. (4.4) and (4.10). From Eq. (4.26) it follows that

$$\varphi(0,q)=0. \tag{4.30}$$

The main contribution to the radiation field $f(\tau,\xi)$ at large τ and ξ comes from $\varphi(\tau,q)$ in the resonant region, where q is sufficiently close to $q_r=\pm k/\kappa$; here, k satisfies the resonant equation (4.11). As far as $\Omega(q_r)=0$, we can replace $\Omega(q)$ in the resonant region by

$$\Omega(q)\approx(q-q_r)\Omega_r',\quad q_r=\pm k/\kappa, \tag{4.31}$$

where, according to Eqs. (4.10), (4.8), and (3.19),

$$\Omega_r'=\left[\frac{d\Omega(q)}{dq}\right]_{q=q_r}=(M-1)^{-1}(V_g-M),\quad \Omega_r'>0. \tag{4.32}$$

Equation (4.28) in the resonant region can be replaced by

$$i\partial_\tau\varphi_r(\tau,q)-\Omega(q)\varphi_r(\tau,q)$$
$$-\frac{q_r}{2\pi}\int_{-\infty}^{\infty}\varphi_0(q_r-q')\varphi_r(\tau,q')dq'$$
$$=L_1(\kappa q_r)\varphi_0(q_r) \tag{4.33}$$

with $\Omega(q)$ from Eq. (4.31). Based on the results, obtained for high-dispersive systems in Refs. [10, 24], we look for the solution to Eq. (4.33) in the form

$$\varphi_r(\tau,q)=\rho\,\frac{\exp[-i(q-q_r)\Omega_r'\tau]-1}{(q-q_r)\Omega_r'}, \tag{4.34}$$

where ρ can be assumed as a constant coefficient (independent of q). This expression is not singular at $q=q_r$, but $\varphi_r(\tau,q_r)\propto\tau$, i.e., $\varphi_r(\tau,q_r)\to\infty$ at $\tau\to\infty$. The width of the resonant region is

$$\Delta q=2\pi/Q_r'\tau; \tag{4.35}$$

thus, $\Delta q\to 0$ at $\tau\to\infty$. Due to that, at large τ,

$$\int_{-\infty}^{\infty}\varphi_0(q_r-q')\varphi_r(\tau,q')dq'\approx\varphi_0(0)\int_{-\infty}^{\infty}\varphi_r(\tau,q')dq'$$
$$=-i\pi\rho\varphi_0(0)/\Omega_r'. \tag{4.36}$$

Substituting Eq. (4.34) into Eq. (4.33) and taking into account Eqs. (4.31) and (4.36), we obtain

$$\rho=\left[1+i\,\frac{q_r\varphi_0(0)}{2\Omega_r'}\right]^{-1}L_1(\kappa q_r)\varphi_0(q_r). \tag{4.37}$$

Using Eqs. (4.32) and (4.29), we have

$$\frac{q_r\varphi_0(0)}{2\Omega_r'}=\pm 6\,\frac{M-1}{V_g-M}\,\frac{k}{\kappa}. \tag{4.38}$$

From Eqs. (4.13), (3.33), and (3.27) it follows that

$$\frac{k}{\kappa}\approx r(\sigma)\left(\frac{1-\sigma}{6(M-1)}\right)^{1/2}\gg 1. \tag{4.39}$$

Then from Eqs. (4.8) and (4.29) we obtain

$$L_1(\kappa q_r)\approx\pm(k/\kappa)^3, \tag{4.40a}$$

$$\varphi_0(q_r)\approx 24\pi\,\frac{k}{\kappa}\,\exp\left(-\pi\,\frac{k}{\kappa}\right). \tag{4.40b}$$

Now we restore $f_r(\tau,\xi)$ by means of

$$f_r(\tau,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_r(\tau,q)e^{iq\xi}dq=\frac{\rho\,\exp(iq_r\xi)}{\Omega_r'}\,I(\tau,\xi)$$
$$+\text{c.c.}, \tag{4.41}$$

where

$$I(\tau,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp[i\chi(\xi-\Omega_r'\tau)-\exp(i\chi\xi)]}{\chi}\,d\chi. \tag{4.42}$$

As long as the integrand in Eq. (4.42) has no singularity, we can replace its denominator by $\chi-i\delta$, where $\delta\to+0$. Then, straightforward calculations give

$$I(\tau,\xi)=i[\Theta(\xi-\Omega_r'\tau)-\Theta(\xi)], \tag{4.43}$$

where $\Theta(Z)$ is the step function:

$$\Theta(Z)=\begin{cases}1 & (Z>0)\\ 0 & (Z<0).\end{cases} \tag{4.44}$$

Evidently, Eq. (4.43) can be written as

$$I(\tau,\xi)=-i\Theta(\xi)\Theta(\Omega_r'\tau-\xi). \tag{4.45}$$

From Eq. (4.45) it follows that, at $\xi\neq 0$, the function $f_r(\tau,\xi)$ satisfies initial condition (4.26).

Collecting all factors determining Eq. (4.41) and using Eqs. (4.1), (4.2), and (4.5) we finally come to the following asymptotic expression for the soliton radiation:

$$v_r(t,x)\approx 32\pi\,\frac{(M-1)^2}{\lambda(V_g-M)}\left(\frac{k}{\kappa}\right)^4$$
$$\times\exp\left(-\frac{\pi k}{\kappa}\right)\operatorname{Im}\left\{\left(1+i\,\frac{6(M-1)}{V_g-M}\,\frac{k}{\kappa}\right)^{-1}\right.$$
$$\left.\times\exp[ik(x-Mt)]\right\}\Theta(x-Mt)\Theta(V_gt-x). \tag{4.46}$$

It is valid at

$$|x-Mt|\gg\kappa^{-1},\quad |V_g t-x|\gg\kappa^{-1}. \tag{4.47}$$

In accordance with Eq. (4.25), the soliton radiates forward and from Eq. (4.46) it follows that the radiation is located at

$$Mt<x<V_g t, \tag{4.48}$$

in agreement with our initial conditions. The discontinuities in Eq. (4.46) evidently appear due to its asymptotic character at conditions (4.47).

From Eqs. (4.46) and (4.39) and Fig. 1 we see that the amplitude of radiation is exponentially small and it decreases with σ. In particular, as it follows from Eq. (4.18), the radiation vanishes at $\sigma\to 0$. This again disagrees with a prediction following from the fifth-order KdV equation. We also note that the full perturbation theory for shallow water solitons with the assumption $\nu=\lambda$ [4] was developed only for $\sigma=0$. One should expect that at $0<\sigma<1$, when the resonant radiation takes place, this perturbation theory should be somehow modified. This is, however, beyond the scope of the present paper.

The factor in the curly brackets of Eq. (4.46) has, according to Eq. (4.24), different behavior in two cases. Namely,

$$V_g-M\approx 2(M-1)\,\frac{\tanh r}{r}\ll 1\quad [r(\sigma)\sim 1] \tag{4.49}$$

and $V_g-M\sim 1$ at $r(\sigma)\gg 1$ [see Eq. (4.25b)]. Of course, the most important case is Eq. (4.49), because then the radiation is larger. Due to Eqs. (4.49) and (4.39),

$$1+i\,\frac{6(M-1)}{V_g-M}\,\frac{k}{\kappa}\approx i\,\frac{3k}{\kappa}\,\frac{r}{\tanh r}, \tag{4.50}$$

and Eq. (4.46) is reduced to

$$v_r(t,x)\approx-\frac{16\pi}{3\lambda}\,(M-1)\left(\frac{k}{\kappa}\right)^3\exp\left(-\frac{\pi k}{\kappa}\right)\times\cos[k(x-Mt)]\Theta(x-Mt)\Theta(V_g t-x)\quad [r(\sigma)\sim 1]. \tag{4.51}$$

It should also be mentioned that the straightforward account of higher-order terms describing small soliton deformations leads to small renormalizations of u_0 in Eq. (4.7). As a result, we would have corresponding small renormalizations of the second term in the brackets in Eq. (4.37) as well as small corrections to the factor beyond them. This would give insignificant corrections to Eq. (4.46). From this it follows, in particular, that higher-order nonlinear corrections can be neglected as far as the soliton radiation is is considered.

Similar results can be obtained (after a more cumbersome algebra) directly from the GB system, derived in Sec. II.

Equation (3.21) was earlier applied to the problem of the soliton radiation in a two-layer model of stratified fluids [25]. Here, we shall not discuss this model, noting only that neither the approach nor the results of Ref. [25] are similar to ours.

An extension of the above approach to the radiation of nonlinear Schrödinger solitons and a comparison with the perturbation theory developed in Ref. [5] will be considered elsewhere.

V. SOLITON DAMPING CAUSED BY THE RADIATION

The radiation must lead to the slowing down of the soliton and, respectively, to a decrease of its amplitude. To estimate this, we start from the GKdV equation (3.21), writing it in the form

$$v_t+\frac{3}{2}\,\lambda v v_x+i\omega(\hat{k})v=0. \tag{5.1}$$

The last term contains an integral operator, defined by Eq. (2.15b). Multiplying Eq. (5.1) by $v(t,x)$ and integrating, we have

$$\partial_t\int_{-\infty}^{\infty}v^2(t,x)dx+i\int_{-\infty}^{\infty}v(t,x)\omega(\hat{k})v(t,x)dx=0. \tag{5.2}$$

[From Eqs. (4.5), (4.6), and (4.26) it follows that $v(t,x)$ vanishes at $x\to\pm\infty$.] From Eqs. (2.15b) and (4.12), one can see that $i\omega(\hat{k})$ is an anti-Hermitian operator. [Due to that expansion (3.22) contains only odd derivatives with real coefficients.] From this we conclude that the second term in Eq. (5.2) is pure imaginary. The first term is, evidently, real. Then each term in Eq. (5.2) must be equal to zero. In particular,

$$\partial_t\int_{-\infty}^{\infty}v^2(t,x)dx=0. \tag{5.3}$$

Substituting here $v\approx v_s+v_r$, where v_s and v_r are the soliton and its radiation (in the following, the soliton deformation is neglected), we have

$$\partial_t\int_{-\infty}^{\infty}v_s^2(t,x)dx\approx-\partial_t\int_{-\infty}^{\infty}\langle v_r^2(t,x)\rangle, \tag{5.4}$$

where the angular brackets mean the averaging over fast oscillations in Eq. (4.46). Using Eqs. (4.1), (4.2), and (4.6), we obtain

$$\int_{-\infty}^{\infty}v_s^2(t,x)dx=\frac{32(M-1)^2}{3\kappa\lambda^2}. \tag{5.5}$$

From Eqs. (4.46) and (4.39) we have

$$\int_{-\infty}^{\infty}\langle v_r^2(t,x)\rangle dx=\frac{64\pi^2(1-\sigma)^4}{81\lambda^2(V_g-M)}\times r^8\left[1+\frac{6(M-1)(1-\sigma)r^2}{(V_g-M)^2}\right]^{-1}\times\exp\left[-\pi\left(\frac{2(1-\sigma)}{3(M-1)}\right)^{1/2}r\right]t. \tag{5.6}$$

Here, $r=r(\sigma,M)$ is the root of Eq. (4.13) and M is a (slow) function of t. Substituting Eqs. (5.5) and (5.6) into Eq. (5.4), we obtain an equation describing the slowing down and the decrease of the soliton amplitude

$$\frac{d(M-1)}{dt}\approx-\frac{4\pi^2}{81}\left(\frac{3}{\nu}\right)^{1/2}\frac{(1-\sigma)^3r^8}{(V_g-M)}\left(\frac{2(1-\sigma)}{3(1-M)}\right)^{1/2}\times\left[1+\frac{6(M-1)(1-\sigma)r^2}{(V_g-M)^2}\right]^{-1}\times\exp\left[-\pi\left(\frac{2(1-\sigma)}{3(M-1)}\right)^{1/2}r\right]. \quad (5.7)$$

Taking into account Eq. (3.28), we see that Eq. (5.7) is exponentially small, which justifies the adiabatic approach used in this paper (see also below).

Equation (5.7) can be approximately integrated by the approach developed in Ref. [26]. Introducing a new dependent variable

$$y(t)=\pi r\left(\frac{2(1-\sigma)}{3(M(t)-1)}\right)^{1/2}, \quad (5.8)$$

and denoting $y_0=y(0)$, we see that

$$y(t)>y_0\gg 1. \quad (5.9)$$

Now, for simplicity, consider the most important case (4.49). Then from Eqs. (5.7) and (5.8) it follows that

$$y^{-4}e^y\frac{dy}{dt}\approx\frac{(1-\sigma)r^2\tanh r}{27\pi\sqrt{3\nu}}. \quad (5.10)$$

Integrating this equation with an account of Eq. (5.9), we approximately have (cf. Ref. [26])

$$y(t)\approx y_0[1+y_0^{-1}\ln(1+t/t_{\mathrm{ch}})]. \quad (5.11)$$

Here, t_{ch} is a characteristic time, defined by

$$t_{\mathrm{ch}}\approx\frac{27\pi\sqrt{3\nu}}{(1-\sigma)r^2\tanh r}y_0^{-4}e^{y_0}=\frac{9\pi^4r}{\tanh r}y_0^{-7}e^{-y_0}\tau_s, \quad (5.12)$$

where τ_s is the ''soliton time''

$$\tau_s=[\kappa(M-1)]_{t=0}^{-1}. \quad (5.13)$$

It is the time during which the soliton passes a distance equal to its width. At the above made assumptions, $t_{\mathrm{ch}}/\tau_s\gg 1$.

From Eqs. (5.8) and (5.11) we have

$$M(t)-1\approx\frac{M_0-1}{[1+y_0^{-1}\ln(1+t/t_{\mathrm{ch}})]^2}, \quad (5.14)$$

where y_0 is given by Eq. (5.8) at $t=0$.

At $t\sim t_{\mathrm{ch}}$ (or larger) the wave train becomes inhomogeneous, because its local amplitude depends on the soliton amplitude $a\propto M-1$ at the moment of radiation. Therefore, the characteristic length of the radiated wave train is $L_{\mathrm{ch}}\sim(V_g-M)t_{\mathrm{ch}}$.

VI. CONCLUSION

Starting from the basic equations of gravity-capillary waves in ideal incompressible fluids, we derived by a multiscale method the generalized Boussinesq system and the generalized KdV equation for shallow-water waves in which high-order dispersive effects of any order of ν [from Eq. (2.6)] are taken into account, but the considered nonlinear terms are of order λ, as in the classical Boussinesq and KdV equations. In linear approximation, from the GB and GKdV equations, the exact dispersive law for the gravity-capillary waves follows. Expanding a dispersive term (which contains an integral operator) in powers of dispersive parameter ν, one comes to the fifth- and higher-order KdV-type equations. The solitons, described by the GKdV (and GB) equation, radiate if $M>1$ and $0<\sigma<1$. Otherwise, they are steady.

In fact, the effect of soliton radiation follows also from the fifth-order KdV equation. However, the radiation wave numbers, phase and group velocities as well as amplitudes, following from the GKdV equation, essentially differ from those predicted by the fifth-order KdV equation. Therefore, the correct description of the radiation can be achieved only by an account of the dispersion to all orders of ν, i.e., on the basis of GKdV or GB equations. On the other hand, the soliton profile, following from GKdV and GB equations coincides, with a sufficient accuracy, with the classical shallow-water soliton described by the regular KdV equation. All higher-order nonlinear corrections to shallow-water solitons and their radiation can be neglected.

Finally we note that the present approach can be used to derive generalized Kadomtzev-Petviashvili and nonlinear Schrödinger equations, taking into account higher-order dispersive effects and permitting one to develop a consistent theory of the resonant soliton radiation in corresponding systems. These problems are under investigation.

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